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LETTER TO THE EDITOR

Group integrals in the lattice QCD with Susskind fermions

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Abstract. Some group integrals which appear in the lattice QCD with Susskind fermions for zero and finite temperatures in the framework of strong coupling approximation are calculated.

Recently, a substantial amount of work has been devoted to the investigation of lattice gauge theories with dynamical fermions both at zero and finite temperatures. A considerable number of Monte Carlo studies, with the help of computers, have been undertaken (see, e.g., Bunk *et al* 1986).

At the same time the analytical calculations in the framework of the strong coupling approximation for understanding qualitatively the long-distance structure of lattice theories with fermions have been carried out (Kawamoto and Smit 1981, Jolicoeur *et al* 1984, Azakov and Aliev 1987, Damgaard *et al* 1984, Fäldt and Petersson 1986).

Dynamical fermions on the lattice can be introduced in different ways if one is aware of the well known fermion spacing doubling problem arising when fermions are considered on the lattice. One of these ways is to use the so-called Susskind (or staggered) lattice fermions (Susskind 1977, Sharatchandra *et al* 1981, Gliozzi 1982) when only one component for each colour Grassmann field $\bar{\chi}_i$ and χ_i are placed at each site of the lattice. On the d-dimensional lattice 2^d-fold degeneracy is then used to construct the $2^{[d/2]}$-component Dirac spinors and the residual $2^{[d/2]}$-fold degeneracy is interpreted as flavour degrees of freedom.

In analytical investigations of the lattice QCD with the Susskind fermions in the strong coupling expansion framework it is necessary to calculate group integrals of the following form (Rossi and Wolff 1984, Azakov and Aliev 1987):

$$I = \int \mathrm{d}U \exp(\bar{\varphi}U\chi + \bar{\chi}U^{+}\varphi) \tag{1a}$$

$$J_{ik} = \int \mathrm{d}U\, U_{ik} \exp(\bar{\varphi}U\chi + \bar{\chi}U^{+}\varphi)$$

$$J^{+}_{ik} = \int \mathrm{d}U\, U^{+}_{ik} \exp(\bar{\varphi}U\chi + \bar{\chi}U^{+}\varphi) \tag{1b}$$

and the integrals

$$R = \int \mathrm{d}U \det(X_N \mathbb{1}_n + \alpha U + \beta U^{+}) \tag{2a}$$

$$L \equiv \int \mathrm{d}U \operatorname{Tr} U \det(X_N \mathbb{1}_n + \alpha U + \beta U^{+}) \tag{2b}$$

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if finite temperatures are considered (Damgaard *et al* 1984, Fäldt and Petersson 1986, Fäldt and Ohlsson 1987). Here $\mathrm{d}U$ is the Haar measure on the gauge colour group (usually $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ groups are considered as gauge groups) φ_i, $\bar{\varphi}_i$, χ_i and $\bar{\chi}_i$ are one-component per colour index i Grassmann variables (colour indices in all expressions are not shown explicitly), $\mathbb{1}_n$ is a unit $n\times n$ matrix, and X_N, α and β are ordinary numbers.

The integral (1a) has been calculated by Rossi and Wolff (1984) (see also Azakov and Aliev 1988):

$$I=\sum_{k=0}^{n}\alpha_k(-\bar{\chi}\cdot\chi\ \bar{\varphi}\cdot\varphi)^k+\frac{\varkappa}{n!}[(\bar{\chi}\cdot\varphi)^n+(\bar{\varphi}\cdot\chi)^n] \tag{3}$$

where

$$\alpha_k=(n-k)!/n!k! \tag{4}$$

and

$$\varkappa=\begin{cases}0 & \text{for the U}(n)\text{ group}\\ 1 & \text{for the SU}(n)\text{ group.}\end{cases}$$

The integrals in (1b) can then easily be calculated if one adds to the exponent in the integrand of I source terms $C_{ik}\bar{U}_{ik}$, $\bar{C}_{ik}U^{+}_{ik}$ with the ordinary c-number sources C_{ik} and $\bar{C}_{ik}$, and then differentiate with respect to them. So we have

$$J_{ik}=\sum_{m=1}^{n}\alpha_m m(-\bar{\chi}\cdot\chi\bar{\varphi}\cdot\varphi)^{m-1}\bar{\chi}_k\varphi_i+\frac{\varkappa}{n!(n-1)!}\varepsilon_{ii_2\ldots i_n}\varepsilon_{kj_2\ldots j_n}\bar{\varphi}_{i_n}\ldots\bar{\varphi}_{i_2}\chi_{j_2}\ldots\chi_{jn} \tag{5a}$$

$$J^{+}_{ik}=\sum_{m=1}^{n}\alpha_m m(-\bar{\chi}\cdot\chi\bar{\varphi}\cdot\varphi)^{m-1}\bar{\varphi}_k\chi_i+\frac{\varkappa}{n!(n-1)!}\varepsilon_{ii_2\ldots i_n}\varepsilon_{kj_2\ldots j_n}\bar{\chi}_{i_n}\ldots\bar{\chi}_{i_2}\varphi_{j_2}\ldots\varphi_{j_n}. \tag{5b}$$

The group integrals (2a) and (2b) have been calculated in Fäldt and Petersson (1986) but they can be calculated much easier if one uses the particular cases of (3) and (5a):

$$\int\mathrm{d}U\exp(A\bar{\chi}U\chi+B\bar{\chi}U^{+}\chi)=\sum_{k=0}^{[n/2]}\alpha_k(-AB)^k(\bar{\chi}\cdot\chi)^{2k}+\frac{\varkappa}{n!}(A^n+B^n)(\bar{\chi}\cdot\chi)^n \tag{6}$$

$$\int\mathrm{d}U\,U_{ik}\exp(A\bar{\chi}U\chi+B\bar{\chi}U^{+}\chi)=\sum_{k=1}^{[n/2]}\alpha_k(-AB)^{k-1}k(\bar{\chi}\chi)^{2k-2}B\bar{\chi}_k\chi_i+\frac{\varkappa}{n!}A^n[(\bar{\chi}\chi)^{n-1}\delta_{ik}+(n-1)(\bar{\chi}\chi)^{n-2}\chi_i\bar{\chi}_k] \tag{7}$$

$$\int\mathrm{d}U\,\mathrm{Tr}\,U\exp(A\bar{\chi}U\chi+B\bar{\chi}U^{+}\chi)=\sum_{k=1}^{[n/2]}\alpha_k(-AB)^{k-1}k(\bar{\chi}\cdot\chi)^{2k-1}+\varkappa\frac{(\bar{\chi}\cdot\chi)^{n-1}}{n!}A^{n-1} \tag{8}$$

and the well known formulae for the integrals over the Grassmann variables

$$\int \mathrm{d}\chi\, \mathrm{d}\bar{\chi}\, \exp(\bar{\chi}M\chi) = \det M \tag{9}$$

$$\int \mathrm{d}\chi\, \mathrm{d}\bar{\chi}\, \exp(\bar{\chi}\cdot\chi)(\bar{\chi}\cdot\chi)^m = n!/(n-m)! \tag{10}$$

where $\mathrm{d}\chi\,\mathrm{d}\bar{\chi} = \Pi_{i=1}^n \mathrm{d}\chi_i\,\mathrm{d}\bar{\chi}_i$, and A and B are ordinary numbers.

Actually, using (6), (9) and (10), we obtain

$$\begin{aligned}\int \mathrm{d}U \det(\mathbb{1}_n + AU + BU^+) &= \int \mathrm{d}\chi\,\mathrm{d}\bar{\chi}\,\exp(\bar{\chi}\cdot\chi)\,\mathrm{d}U\,\exp(A\bar{\chi}U\chi + B\bar{\chi}U\chi)\\ &= \sum_{k=0}^{[n/2]} \binom{n-k}{k}(-AB)^k + \varkappa(A^n + B^n)\end{aligned} \tag{11}$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Thus

$$R = \sum_{k=0}^{[n/2]} \binom{n-k}{k}(-\alpha\beta)^k X_N^{n-2k} + \varkappa(\alpha^n + \beta^n). \tag{12}$$

Now using again (7), (9) and (10), we have

$$\begin{aligned}\int \mathrm{d}U\,\mathrm{Tr}\,U \det(\mathbb{1}_n + AU + BU^+) &= \int \mathrm{d}\chi\,\mathrm{d}\bar{\chi}\,\exp(\bar{\chi}\cdot\chi)\int \mathrm{d}U\,\mathrm{Tr}\,U\,\exp(A\bar{\chi}U\chi + B\bar{\chi}U\chi)\\ &= B\sum_{k=1}^{[n/2]} \frac{(n-k)!(-AB)^{k-1}}{(k-1)!(n-2k+1)!} + \varkappa A^{n-1}.\end{aligned} \tag{13}$$

Thus

$$L = \beta \sum_{k=0}^{[(n-1)/2]} \binom{n-k+1}{k}(-\alpha\beta)^k X_N^{n-2k+1} + \varkappa X_N \alpha^{n-1}. \tag{14}$$

Our expressions (12) and (14) for R and L coincide with expressions (B.7) and (C.17) respectively, in Fäldt and Petersson (1986). This follows from the formula

$$\sum_{k=0}^{[n/2]} \binom{n-k}{k}(-\xi)^k = \frac{(\xi/x)^{n+1} - x^{n+1}}{\xi/x - x} \tag{15}$$

where $\xi/x + x = 1$.

Equation (15) can be obtained by differentiating with respect to ξ the formula

$$n\sum_{k=0}^{[n/l]} \frac{(n-k-1)!}{k!(n-2k)!}(-\xi)^k = [\tfrac{1}{2} + (\tfrac{1}{4} - \xi)^{1/2}]^n + [\tfrac{1}{2} - (\tfrac{1}{4} - \xi)^{1/2}]^n \tag{16}$$

which can be found, for example, in Prudnikov *et al* (1986), formula (4.2.3.26).

In a particularly important special case, when $\alpha = \beta = 1$ and $X_N = 2\cosh N\zeta$, from our expression (12) we can easily get (B.9) in Fäldt and Petersson (1986) using the formula

$$\sinh(n+1)x = \sinh x \sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} (2\cosh x)^{n-2k}$$

(see, e.g., Gradshteyn and Ryzhik (1980) formula 1.331.2).

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